



## ON THE SPECTRUMS OF SOME CLASS OF SELFADJOINT SINGULAR DIFFERENTIAL OPERATORS

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**ABSTRACT.** In this work, based on the Everitt-Zettl and Calkin-Gorbachuk methods in terms of boundary values all selfadjoint extensions of the minimal operator generated by some linear singular multipoint symmetric differential-operator expression for first order in the direct sum of Hilbert spaces of vector-functions on the right semi-axis are described. Later structure of the spectrum of these extensions is investigated.

### 1. INTRODUCTION

The general theory of selfadjoint extensions of symmetric operators in any Hilbert space and their spectral theory have deeply been investigated by many mathematicians (for example, see [1-6]). Applications of this theory to two point differential operators in Hilbert space of functions are continued today even. It is known that for the existence of selfadjoint extension of the any linear closed densely defined symmetric operator  $B$  in a Hilbert space, the necessary and sufficient condition is an equality of deficiency indices  $m(B) = n(B)$ , where  $m(B) = \dimker(B^* + i)$ ,  $n(B) = \dimker(B^* - i)$  [1]. The table is changed in the multipoint case in the following sense. Let  $L_1$  and  $L_2$  be minimal operators generated by the linear differential expression  $l(u) = i\frac{d}{dt}$  and  $m(u) = -i\frac{d}{dt}$  in the Hilbert space of functions  $L^2(a, +\infty)$  and  $L^2(b, +\infty)$ ,  $a, b \in \mathbb{R}$ , respectively. Consider the deficiency indices of  $L_1$  and  $L_2$ . In this case it is known that  $(m(L_1), n(L_1)) = (1, 0)$ ,  $(m(L_2), n(L_2)) = (0, 1)$ . Consequently,  $L_1$  and  $L_2$  are maximal symmetric operators, but are not selfadjoint [1]. However, direct sum  $L = L_1 \oplus L_2$  of operators  $L_1$  and  $L_2$  in  $L^2(a, +\infty) \oplus L^2(b, +\infty)$  of Hilbert spaces have an equal defect numbers  $(1, 1)$ . Then by the general theory [1] it has a selfadjoint extension. On the other hand it can be easily shown in the form that

$$u_2(b) = e^{i\varphi}u_1(a), \varphi \in [0, 2\pi), u = (u_1, u_2), \quad u_1 \in D(L_1^*), u_2 \in D(L_2^*).$$

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Note that in the multi interval linear ordinary differential expression case the deficiency indices may be different for each interval, but equal in the direct sum Hilbert spaces from the different intervals. The selfadjoint extension theory for any order linear ordinary differential expression case is known from famous work of W.N. Everitt and A. Zettl [7] for any number of finite and infinite intervals of real-axis. This theory is based on the Glazman-Krein-Naimark Theorem. In formations on the selfadjoint extensions, the direct and complete characterizations for the Sturm-Liouville differential expression in finite or infinite interval with interior points or endpoints singularities can be found in the significant monograph of A.Zettl [8].

Lastly, note that many problems arising in the modeling of processes of multi-particle quantum mechanics, quantum field theory, in the physics of rigid bodies and etc. support to study selfadjoint extensions of symmetric differential operators in direct sum of Hilbert spaces (see [8] and references in it).

In this work in second section, by the methods of Everitt-Zettl and Calkin-Gorbachuk Theories, all selfadjoint extensions of the minimal operator generated by linear multipoint singular symmetric differential-operator expression of first order in the direct sum of Hilbert spaces  $L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))$ , described, where  $H_1, H_2$  are a separable Hilbert spaces with condition  $0 < \dim H_1 = \dim H_2$  and  $a, b \in \mathbb{R}$ , in terms of boundary values. In third section the spectrum of such extensions is researched.

## 2. DESCRIPTION OF SELFADJOINT EXTENSIONS

Let  $H_1, H_2$  be separable Hilbert space with  $0 < \dim H_1 = \dim H_2 \leq \infty$  and  $a, b \in \mathbb{R}$ ,  $a < b$ . In the Hilbert space  $L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))$  of vector-functions considers the following linear multipoint differential-operator expressions

$$\begin{aligned} l(u) &= (l_1(u_1), l_2(u_2)), \quad \text{where } u = (u_1, u_2), \\ l_1(u_1) &= iu_1'(t) + A_1 u_1(t), \quad t \in (a, +\infty), \\ l_2(u_2) &= -iu_2'(t) + A_2 u_2(t), \quad t \in (b, +\infty), \end{aligned}$$

where  $A_k : D(A_k) \subset H_k \rightarrow H_k$  are linear selfadjoint operators in  $H_k$ ,  $k = 1, 2$ . In the linear manifold  $D(A_k) \subset H_k$  introduces the inner product in form

$$(f, g)_{k,+} = (A_k f, A_k g)_{H_k} + (f, g)_{H_k}, \quad f, g \in D(A_k), \quad k = 1, 2.$$

For  $k = 1, 2$   $D(A_k)$  is a Hilbert space under the positive norm  $\|\cdot\|_{k,+}$  respect to the Hilbert space  $H_k$ . It is denoted by  $H_{k,+}$ ,  $k = 1, 2$ . Denote the Hilbert spaces with the negative norm by  $H_{k,-}$ ,  $k = 1, 2$ . It is clear that an operator  $A_k$  is continuous from  $H_{k,+}$  to  $H_k$  and that its adjoint operator  $\tilde{A}_k : H_k \rightarrow H_{k,-}$  is an extension of the operator  $A_k$ ,  $k = 1, 2$ . On the other hand, the operator  $\hat{A}_k : H_k \subset H_{k,-} \rightarrow H_{k,-}$   $k = 1, 2$  are linear selfadjoint operators.

In  $L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))$  define

$$\tilde{l}(u) = (\tilde{l}_1(u_1), \tilde{l}_2(u_2)), \tag{2.1}$$

where  $u = (u_1, u_2)$ ,  $\tilde{l}_1(u_1) = iu_1'(t) + \tilde{A}_1 u_1(t)$ ,  $t \in (a, +\infty)$ ,  $\tilde{l}_2(u_2) = -iu_2'(t) + \tilde{A}_2 u_2(t)$ ,  $t \in (b, +\infty)$ .

The minimal  $L_{10}$  ( $L_{20}$ ) and maximal  $L_1$  ( $L_2$ ) operators generated by differential-operator expression  $\tilde{l}_1(\cdot)$  ( $\tilde{l}_2(\cdot)$ ) in  $L^2(H_1, (a, +\infty))$  ( $L^2(H_2, (b, +\infty))$ ) have been investigated in [5] and here established that the minimal operator  $L_{10}$  ( $L_{20}$ ) is not selfadjoint in  $L^2(H_1, (a, +\infty))$  ( $L^2(H_2, (b, +\infty))$ ). The operators defined by  $L_0 = L_{10} \oplus L_{20}$  and  $L = L_1 \oplus L_2$  in  $L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))$  are called minimal and maximal (multipoint) operators generated by the differential expression (2.1), respectively. Note that the operator  $L_0$  is a symmetric operator in  $L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))$ . On the other hand, it is clear that

$$m(L_{10}) = 0, \quad n(L_{10}) = \dim H_1,$$

$$m(L_{20}) = \dim H_2, \quad n(L_{20}) = 0.$$

Consequently,  $m(L_0) = \dim H_2 > 0$ ,  $n(L_0) = \dim H_1 > 0$ . So the minimal operator  $L_0$  in  $L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))$  has a selfadjoint extension [1].

In first note that the following proposition which validity of this clear can be easily proved.

**Proposition 2.1.** *Let us  $L_{n0}$ ,  $M_{n0}$  and  $K_{n0}$  be minimal operators generated by linear differential expressions*

$$l_n(u_n) = (-1)^{n-1} iu_n'(t) + A_n u_n(t), \quad t \in (a_n, +\infty), \quad a_n \in \mathbb{R},$$

$$m_n(u_n) = iu_n'(t) + B_n u_n(t), \quad t \in (-\infty, b_n), \quad b_n \in \mathbb{R},$$

$$k_n(u_n) = iu_n'(t) + C_n u_n(t), \quad t \in (c_n, +\infty), \quad c_n \in \mathbb{R}, \quad n = 1, 2, \dots, m,$$

where  $A_n : D(A_n) \subset H_n \rightarrow H_n$ ,  $B_n : D(B_n) \subset H_n \rightarrow H_n$ ,  $C_n : D(C_n) \subset H_n \rightarrow H_n$  are linear selfadjoint operators in the Hilbert space of vector-functions  $L^2(H_n, (a_n, +\infty))$ ,  $L^2(H_n, (-\infty, b_n))$  and  $L^2(H_n, (c_n, +\infty))$ ,  $n = 1, 2, \dots, m$ , respectively and  $\dim H_1 = \dim H_2 = \dots = \dim H_m \leq \infty$ .

In this case:

- (1) For any  $n = 1, 2, \dots, m$  the minimal operators  $L_{n0}$ ,  $M_{n0}$  and  $K_{n0}$  have not selfadjoint extensions in  $L^2(H_n, (a_n, +\infty))$ ,  $L^2(H_n, (-\infty, b_n))$  and  $L^2(H_n, (c_n, +\infty))$ ,  $n = 1, 2, \dots, m$ , respectively (see[5]).
- (2) If  $m$  is a even integer number, then the multipoint minimal operator  $L_0 = \bigoplus_{n=1}^m L_{n0}$  have a selfadjoint extension in  $\bigoplus_{n=1}^m L^2(H_n, (a_n, +\infty))$ ;
- (3) If  $m$  is a odd integer number, then the multipoint minimal operator  $L_0 = \bigoplus_{n=1}^m L_{n0}$  is not selfadjoint extension in  $\bigoplus_{n=1}^m L^2(H_n, (a_n, +\infty))$ ;
- (4) The multipoint minimal operator  $L_0 = M_{10} \oplus K_{10} \oplus M_{20} \oplus K_{20} \oplus \dots \oplus M_{m0} \oplus K_{m0}$  is a selfadjoint operator in  $\bigoplus_{n=1}^m (L^2(H_n, (-\infty, b_n)) \oplus L^2(H_n, (c_n, +\infty)))$ ;
- (5) The multipoint minimal operator  $L_0 = M_{10} \oplus K_{10} \oplus M_{20} \oplus K_{20} \oplus \dots \oplus M_{(m-1)0} \oplus K_{(m-1)0} \oplus M_{m0}$  is not selfadjoint operator in  $\bigoplus_{n=1}^{m-1} (L^2(H_n, (-\infty, b_n))) \oplus L^2(H_m, (c_m, +\infty)) \oplus L^2(H_m, (-\infty, b_m))$ ;

- (6) *The multipoint minimal operator  $L_0 = M_{10} \oplus K_{10} \oplus M_{20} \oplus K_{20} \oplus \dots \oplus M_{(m-1)0} \oplus K_{(m-1)0} \oplus K_{m0}$  is not selfadjoint operator in  $\bigoplus_{n=1}^{m-1} (L^2(H_n, (-\infty, b_n))) \oplus L^2(H_n, (c_n, +\infty)) \oplus L^2(H_m, (c_m, +\infty))$ .*

In this section all selfadjoint extensions of the minimal operator  $L_0$  generated by linear multipoint symmetric differential-operator expression of first order (2.1) in the direct sum of Hilbert spaces  $L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))$  in terms of the boundary values will be described. Note that in the Calkin-Gorbachuk theory of selfadjoint extensions of the linear symmetric densely defined closed operators co-called "space of boundary values" has an important role [3,4].

Firstly, let us recall their definition.

**Definition 2.1.** [3] *Let  $T : D(T) \subset H \rightarrow H$  be a closed densely defined symmetric operator in the Hilbert space  $H$ , having equal finite or infinite deficiency indices. A triplet  $(\mathfrak{H}, \gamma_1, \gamma_2)$ , where  $\mathfrak{H}$  is a Hilbert space,  $\gamma_1$  and  $\gamma_2$  are linear mappings of  $D(T^*)$  into  $\mathfrak{H}$ , is called a space of boundary values for the operator  $T$  if for any  $f, g \in D(T^*)$*

$$(T^*f, g)_H - (f, T^*g)_H = (\gamma_1(f), \gamma_2(g))_{\mathfrak{H}} - (\gamma_2(f), \gamma_1(g))_{\mathfrak{H}},$$

while for any  $F_1, F_2 \in \mathfrak{H}$ , there exists an element  $f \in D(T^*)$ , such that  $\gamma_1(f) = F_1$  and  $\gamma_2(f) = F_2$ .

Note that any symmetric operator with equal deficiency indices has at least one space of boundary values [3].

Since  $H_1, H_2$  are separable Hilbert spaces and  $\dim H_1 = \dim H_2$ , then it is known that there exist a isometric isomorphism  $V : H_1 \rightarrow H_2$  such that  $VH_1 = H_2$ .

In this case the following proposition is true.

**Lemma 2.1.** *The triplet  $(H_2, \gamma_1, \gamma_2)$  is a space of boundary values of the minimal operator  $L_0$  in  $L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))$ , where*

$$\begin{aligned} \gamma_1 : D(L_0^*) &\rightarrow H_1, \gamma_1(u) = \frac{1}{i\sqrt{2}} (Vu_1(a) + u_2(b)), u \in D(L_0^*), \\ \gamma_2 : D(L_0^*) &\rightarrow H_1, \gamma_2(u) = \frac{1}{\sqrt{2}} (Vu_1(a) - u_2(b)), u \in D(L_0^*). \end{aligned}$$

*Proof.* For arbitrary  $u = (u_1, u_2)$  and  $v = (v_1, v_2)$  in  $D(L)$  the validity of following equality

$$\begin{aligned} (Lu, v)_{L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))} - (u, Lv)_{L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))} \\ = (\gamma_1(u), \gamma_2(v))_{H_1} - (\gamma_2(u), \gamma_1(v))_{H_1} \end{aligned}$$

can be easily verified. Now for any given elements  $f, g \in H_1$ , we will find the function  $u = (u_1, u_2) \in D(L)$  such that

$$\gamma_1(u) = \frac{1}{i\sqrt{2}} (Vu_1(a) + u_2(b)) = f \quad \text{and} \quad \gamma_2(u) = \frac{1}{\sqrt{2}} (Vu_1(a) - u_2(b)) = g$$

that is,

$$u_1(a) = V^{-1}(if + g)/\sqrt{2} \quad \text{and} \quad u_2(b) = (if - g)/\sqrt{2}.$$

If we choose these functions  $u_1(t), u_2(t)$  in form

$$u_1(t) = e^{(a-t)/2} V^{-1}(if + g)/\sqrt{2}, \quad t > a,$$

$$u_2(t) = e^{(b-t)/2} (if - g)/\sqrt{2}, \quad t > b,$$

then it is clear that  $(u_1, u_2) \in D(L)$  and  $\gamma_1(u) = f, \gamma_2(u) = g$ .  $\square$

Furthermore, using the method in [1, 3] the following result can be deduced.

**Theorem 2.2.** *If  $\tilde{L}$  is a selfadjoint extension of the minimal operator  $L_0$  in  $L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))$ , then it generates by differential-operator expression (2.1) and boundary condition*

$$u_2(b) = WVu_1(a)$$

where  $W : H_2 \rightarrow H_2$  is a unitary operator. Moreover, the unitary operator  $W$  is determined uniquely by the extension  $\tilde{L}$ ; i.e.,  $\tilde{L} = L_W$  and vice versa.

**Remark 2.1.** *With similar ideas the selfadjoint extensions of minimal operator generated by multipoint differential-operator expression in  $\bigoplus_{p=1}^n L^2(H_p, (a_p, \infty)) \oplus \bigoplus_{j=1}^k L^2(G_j, (b_j, \infty))$  with condition  $0 < \sum_{p=1}^n \dim H_p = \sum_{j=1}^k \dim G_j$ , can be described*

$$l(u) = (l_1(u_1), l_2(u_2), \dots, l_n(u_n); m_1(v_1), m_2(v_2), \dots, m_k(v_k)),$$

where  $u = (u_1, u_2, \dots, u_n; v_1, v_2, \dots, v_k)$ ,

$$l_p(u_p) = iu_p'(t) + A_p u_p(t), \quad t \in (a_p, \infty), \quad p = 1, 2, \dots, n;$$

$$m_j(v_j) = -iv_j'(t) + B_j v_j(t), \quad t \in (b_j, \infty), \quad j = 1, 2, \dots, k,$$

$A_p : D(A_p) \subset H_p \rightarrow H_p$  and  $B_j : D(B_j) \subset G_j \rightarrow G_j$  are linear selfadjoint operators in Hilbert spaces  $H_k, p = 1, 2, \dots, n$  and  $G_j, j = 1, 2, \dots, k$ , respectively.

### 3. THE SPECTRUM OF THE NORMAL EXTENSIONS

In this section the structure of the spectrum of the selfadjoint extension  $L_W$  in  $L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))$  will be investigated.

First, we will prove the following result.

**Theorem 3.1.** *The point spectrum of selfadjoint extension  $L_W$  is empty, i.e.,*

$$\sigma_p(L_W) = \emptyset.$$

*Proof.* Let us consider the following problem

$$\begin{aligned}\tilde{l}(u) &= \lambda u(t), \lambda \in \mathbb{R}, \\ u_2(b) &= WVu_1(a),\end{aligned}$$

where  $W : H_2 \rightarrow H_2$  is a unitary operator. Then

$$(\tilde{l}_1(u_1), \tilde{l}_2(u_2)) = \lambda(u_1, u_2), \quad u_2(b) = WVu_1(a),$$

and we have

$$\begin{aligned}\tilde{l}_1(u_1) &= iu_1'(t) + \tilde{A}_1 u_1(t) = \lambda u_1(t), \quad t \in (a, +\infty), \\ \tilde{l}_2(u_2) &= -iu_2'(t) + \tilde{A}_2 u_2(t) = \lambda u_2(t), \quad t \in (b, +\infty), \\ u_2(b) &= WVu_1(a), \quad \lambda \in \mathbb{R}.\end{aligned}$$

The general differential solution of this problem is

$$\begin{aligned}u_1(\lambda; t) &= e^{i(\tilde{A}_1 - \lambda)(t-a)} f_\lambda, \quad f_\lambda \in H_1, \quad t \in (a, +\infty), \\ u_2(\lambda; t) &= e^{-i(\tilde{A}_2 - \lambda)(t-b)} g_\lambda, \quad g_\lambda \in H_2, \quad t \in (b, +\infty).\end{aligned}$$

Boundary condition is in form  $g_\lambda = WVf_\lambda$ . In order to show  $u_1(\lambda; t) \in L^2(H_1, (a, +\infty))$  and  $u_2(\lambda; t) \in L^2(H_2, (b, +\infty))$ , the necessary and sufficient conditions are  $f_\lambda = g_\lambda = h_\lambda = 0$ . So for every operator  $W$  we have  $\sigma_p(L_W) = \emptyset$ , where  $\sigma_p(L_W)$  denotes the point spectrum of  $L_W$ .  $\square$

Since residual spectrum of any selfadjoint operator in any Hilbert space is empty, then it is sufficient to investigate the continuous spectrum of the selfadjoint extensions  $L_W$  of the minimal operator  $L_0$ .

Now, we will study continuous spectrum of the selfadjoint extension  $L_W$ , where  $\sigma_c(L_W)$  denotes the continuous spectrum of  $L_W$ .

**Theorem 3.2.** *The continuous spectrum of any selfadjoint extension  $L_W$  is  $\sigma_c(L_W) = \mathbb{R}$ .*

*Proof.* For  $\lambda \in \mathbb{C}$ ,  $\lambda_i = \text{Im}\lambda > 0$ , norm of the resolvent operator of the  $L_W$  is of the form

$$\begin{aligned}\|R_\lambda(L_W)f(t)\|_{L^2(H_1, (a, \infty)) \oplus L^2(H_2, (b, \infty))}^2 &= \|i \int_t^\infty e^{i(\tilde{A}_1 - \lambda)(t-s)} f_1(s) ds\|_{L^2(H_1, (a, \infty))}^2 + \\ &+ \|e^{i(\lambda - \tilde{A}_1)(t-b)} g_\lambda + i \int_b^t e^{i(\lambda - \tilde{A}_1)(t-s)} f_2(s) ds\|_{L^2(H_2, (b, \infty))}^2\end{aligned}$$

where  $f = (f_1, f_2) \in L^2(H_1, (a, \infty)) \oplus L^2(H_2, (b, \infty))$ ,

$$g_\lambda = WV \left( i \int_a^\infty e^{i(\tilde{A}_1 - \lambda)(a-s)} f_1(s) ds \right)$$

and  $R_\lambda(L_W)$  shows the resolvent operator of  $L_W$ . Then, it is clear that for any  $f$  in  $L^2(H_1, (a, \infty)) \oplus L^2(H_2, (b, \infty))$ , the following inequality is true

$$\|R_\lambda(L_W)f(t)\|_{L^2}^2 \geq \|i \int_t^\infty e^{i(\tilde{A}_1 - \lambda)(t-s)} f_1(s) ds\|_{L^2(H_1, (a, \infty))}^2.$$

The vector functions  $f^*(\lambda; t)$  which is of the form  $f^*(\lambda; t) = (e^{i(\bar{A}_1 - \bar{\lambda})t} f, 0)$ ,  $\lambda \in \mathbb{C}$ ,  $\lambda_i = \text{Im}\lambda > 0$ ,  $f \in H_1$  belong to  $L^2(H_1, (a, \infty)) \oplus L^2(H_2, (b, \infty))$ . Indeed,

$$\begin{aligned} \|f^*(\lambda; t)\|_{L^2(H_1, (a, \infty)) \oplus L^2(H_2, (b, \infty))}^2 &= \int_a^\infty \|e^{i(\bar{A}_1 - \bar{\lambda})t} f\|_{H_1}^2 dt \\ &= \int_a^\infty e^{-2\lambda_i t} dt \|f\|_{H_2}^2 = \frac{1}{2\lambda_i} e^{-2\lambda_i a} < \infty. \end{aligned}$$

For such functions  $f^*(\lambda; \cdot)$ , we have

$$\begin{aligned} \|R_\lambda(L_W)f^*(\lambda; t)\|_{L^2(H_1, (a, \infty)) \oplus L^2(H_2, (b, \infty))}^2 &\geq \|i \int_t^\infty e^{i(\bar{A}_1 - \lambda)(t-s)} e^{i(\bar{A}_1 - \bar{\lambda})s} f ds\|_{L^2(H_1, (a, \infty))}^2 \\ &= \left\| \int_t^\infty e^{-i\lambda t} e^{-2\lambda_i s} e^{i\bar{A}_1 t} f ds \right\|_{L^2(H_1, (a, \infty))}^2 \\ &= \|e^{-i\lambda t} e^{i\bar{A}_1 t} \int_t^\infty e^{-2\lambda_i s} f ds\|_{L^2(H_1, (a, \infty))}^2 \\ &= \|e^{-i\lambda t} \int_t^\infty e^{-2\lambda_i s} ds\|_{L^2(H_1, (a, \infty))}^2 \|f\|_{H_1}^2 \\ &= \frac{1}{4\lambda_i^2} \int_a^\infty e^{-2\lambda_i t} dt \|f\|_{H_1}^2 = \frac{1}{8\lambda_i^3} e^{-2\lambda_i a} \|f\|_{H_1}^2. \end{aligned}$$

From this, we obtain

$$\begin{aligned} \|R_\lambda(L_W)f^*(\lambda; \cdot)\|_{L^2(H_1, (a, \infty)) \oplus L^2(H_2, (b, \infty))}^2 &\geq \frac{e^{-\lambda_i a}}{2\sqrt{2}\lambda_i\sqrt{\lambda_i}} \|f\|_{H_1} \\ &= \frac{1}{2\lambda_i} \|f^*(\lambda; \cdot)\|_{L^2(H_1, (a, \infty)) \oplus L^2(H_2, (b, \infty))} \end{aligned}$$

i.e., for  $\lambda_i = \text{Im}\lambda > 0$  and  $f \neq 0$ , the following inequality is valid

$$\frac{\|R_\lambda(L_W)f^*(\lambda; \cdot)\|_{L^2(H_1, (a, \infty)) \oplus L^2(H_2, (b, \infty))}}{\|f^*(\lambda; \cdot)\|_{L^2(H_1, (a, \infty)) \oplus L^2(H_2, (b, \infty))}} \geq \frac{1}{2\lambda_i}$$

is valid. On the other hand, it is clear that

$$\|R_\lambda(L_W)\| \geq \frac{\|R_\lambda(L_W)f^*(\lambda; \cdot)\|_{L^2(H_1, (a, \infty)) \oplus L^2(H_2, (b, \infty))}}{\|f^*(\lambda; \cdot)\|_{L^2(H_1, (a, \infty)) \oplus L^2(H_2, (b, \infty))}}, \quad f \neq 0.$$

Consequently,

$$\|R_\lambda(L_W)\| \geq \frac{1}{2\lambda_i} \text{ for } \lambda \in \mathbb{C}, \quad \lambda_i = \text{Im}\lambda > 0.$$

From last relation it is implies the validity of assertion.  $\square$

**Example 3.1.** Consider the following boundary value problem in  $L^2((0, +\infty) \times (0, 1)) \oplus L^2((0, +\infty) \times (0, 1))$

$$\begin{aligned} i \frac{\partial u(t, x)}{\partial t} - \frac{\partial^2 u(t, x)}{\partial x^2} &= f(t, x), \quad t > 0, x \in [0, 1], \\ i \frac{\partial v(t, x)}{\partial t} - \frac{\partial^2 v(t, x)}{\partial x^2} &= g(t, x), \quad t > 0, x \in [0, 1], \\ u'_x(t, 0) = u'_x(t, 1) = 0, \quad v'_x(t, 0) = v'_x(t, 1) &= 0, \quad t > 0, \\ u(0, x) &= e^{i\varphi} v(0, x), \quad \varphi \in [0, 2\pi). \end{aligned}$$

By using the last theorem, we get that this boundary value problem is continuous and coincides with  $\mathbb{R}$ .

**Remark 3.1.** If we take  $a = b$ , then a differential-operator expression generated by  $l(\cdot)$  can be written in form

$$l(u) = iJu'(t) + Au(t),$$

where  $J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ,  $A := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$  in  $L^2(H_1 \oplus H_2, (a, +\infty))$ . Particularly, the obtained results in this work generalizes some results which have been established in [5].

**Remark 3.2.** The similar problems was considered and analogous results has been obtained in works [9-12].

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