MATEC Web of Conferences **83**, 03005 (2016) DOI: 10.1051/matecconf/20168303005
CSNDD 2016

The Linear Stability of the Responses of Axially Moving Beams Supported by an Intermediate Spring

F. Kostekci[1]

[1]Mechanical Engineering Department, Hitit University, Corum, 19030, Turkey

Abstract. In the present paper, the stability investigation of the linear responses of axially travelling beams supported by an intermediate linear spring is performed numerically by means of searching linear complex eigenvalues. The boundary conditions and the dynamic equation of motion are obtained by means of the Extended Principle of Hamilton for the two-span Euler-Bernoulli beam. Assuming the harmonic solution of the governing differential equations of motion, the dispersion relation is derived. To have a nontrivial solution in conjunction with the dispersion relation, the frequency equation is obtained by letting the determinant of the matrix representing the coefficients of the equations of the boundary conditions be equal to zero. The aim of this contribution is to compute the complex values of the natural frequencies from the nonlinear frequency equation by means of numerical methods for the beam supported by an intermediate spring. The real and imaginary parts of the complex eigenvalues versus the constant axial transport speed curves are depicted in order to show the linear instability. Depending on the signs of the real and imaginary parts of the complex eigenvalues, the boundaries of the instability regions were observed.

1 Introduction

The industrial applications of moving materials classified as axially travelling stringlike, beamlike structures are numerous [1-4] such as magnetic tapes drives, band saws, power transmission belts, robot arms, serpentine belts and aerial cable tramways. The purpose of the current research paper is to investigate the linear instability of the beam. The dynamic equation of motion and

obtained utilizing the Extended Principle of Hamilton for the two-span Euler-Bernoulli beam. The complex mode method [5, 6] is used to obtain the nonlinear frequency equation. The boundaries of the instability regions were determined according to the signs of the real and imaginary parts of the eigenvalues.

2 Formulation of the beam problem

The axially moving continua with an intermediate spring support is illustrated schematically in Fig. 1.

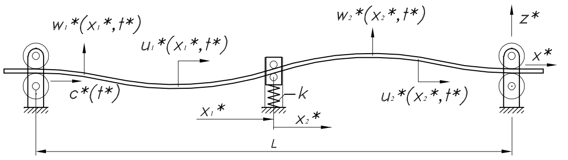

Fig. 1. Geometry of the axially travelling beam.

The equations of motion can be obtained by following the procedure presented in the reference [7] using the Extended Hamilton's Principle. The Euler-Bernoulli beam is moving axially at a slightly fluctuating harmonic velocity c^*. After writing the kinetic and potential energies of the system, the Lagrangian for the axially moving beam can be formulized through the Hamilton's Principle as shown in Eq. (1). In Fig. 1, w_1^*, w_2^* and u_1^* and u_2^* denote respectively the transverse and longitudinal displacements, c^* is the velocity of the axial transport of the beam, t^* denotes time, where the variables with asterisk denote dimensional ones. Properties of the beam include the beam length L, the cross-section area A, I is the cross-sectional inertia of the beam, the mass density ρ and E is the Young's modulus of the beam. k is the translational stiffness of the linear spring. The values of the spatial variable at the boundaries namely x_0^* and x_2^* should be written as 0 and L respectively. It is assumed that the beam has immovable boundaries at the outer ends. The assumption introduces nonlinearity because of the stretching of neutral fibres.

$$
\begin{aligned}
£ &= (1/2)\Big(\int_{x_0^*}^{x_1^*} \rho A((\dot{w}_1^* + w_1^{*\prime} c^*)^2 + (c^* + \dot{u}_1^* + u_1^{*\prime} c^*)^2)^2 \\
& dx^* + \int_{x_1^*}^{x_2^*} \rho A((\dot{w}_2^* + w_2^{*\prime} c^*)^2 + (c^* + \dot{u}_2^* + u_2^{*\prime} c^*)^2)^2 dx^*\Big) \\
& - (1/2)\Big(\int_{x_0^*}^{x_1^*} EA(u_1^{*\prime} + (1/2) w_1^{*\prime})^2 dx^* + \int_{x_0^*}^{x_1^*} EI w_1^{*\prime\prime 2} dx^* \\
& + \int_{x_0^*}^{x_1^*} P(u_1^{*\prime} + (1/2) w_1^{*\prime 2})^2 dx^* \\
& + \int_{x_1^*}^{x_2^*} EA(u_2^{*\prime} + (1/2) w_2^{*\prime})^2 dx^* + \int_{x_1^*}^{x_2^*} EI w_2^{*\prime\prime 2} dx^* \\
& + \int_{x_1^*}^{x_2^*} P(u_2^{*\prime} + (1/2) w_2^{*\prime 2})^2 dx^* \\
& + (1/2) k\, w_1^{*2}(x_1^*)\Big)
\end{aligned} \tag{1}
$$

where P is the total tensional force, which is composed of an applied artificial pre-tension and the tension caused by the extensional and flexural deformations of the centreline, in the axial direction.

2.1 Dimensionless formulation

For convenience, the equations and all accompanying boundary conditions are presented in dimensionless form for the two sides. The following non-dimensional variables and parameters introduced in Ref. [8] are employed.

$$
\begin{aligned}
& w_1 = w_1^*/L, w_2 = w_2^*/L, u_1 = u_1^*/L, u_2 = u_2^*/L \\
& x_0 = x_0^*/L, x_1 = x_1^*/L, x_2 = x_2^*/L, t = \gamma t^* \\
& c = c^*/\sqrt{P/\rho A}, v_b^2 = EA/P, \bar{v}_f^2 = EI/PL^2 \\
& \gamma = (1/L)\sqrt{P/\rho A}
\end{aligned} \tag{2}
$$

In addition to those given in Eq. 2, the dimensionless pan-wise variable, which represents a dimensionless location constant of the intermediate support and may take a value bigger than 0 and smaller than 1, is introduced: $x_1 = \eta$. Dimensionless parameter v_b presents the longitudinal rigidity parameter, and $\bar{v}_f$ denotes the flexural rigidity parameter. The axial velocity is converted to dimensionless quantity by dividing it with the critical velocity. Substituting Eq. 2 in the governing equation of the motion, performing required operations and adding damping term ($\bar{\mu}$) yield the dimensionless equations of motion and relevant non-dimensional boundary conditions.

$$
\begin{aligned}
& \ddot{w}_1 + 2\dot{w}_1' c + w_1' \dot{v} + w_1''(c^2 - 1) + \bar{v}_f^2 w_1^{iv} + \\
& \bar{\mu}(\dot{w}_1 + c\, w_1') = v_b^2 (1/2)\Big(\int_0^\eta w_1'^2 dx + \int_\eta^1 w_2'^2 dx\Big) w_1''
\end{aligned}
$$

$$
\begin{aligned}
& \ddot{w}_2 + 2\dot{w}_2' c + w_2' \dot{v} + w_2''(c^2 - 1) + \bar{v}_f^2 w_2^{iv} + \\
& \bar{\mu}(\dot{w}_2 + c\, w_2') = v_b^2 (1/2)\Big(\int_0^\eta w_1'^2 dx + \int_\eta^1 w_2'^2 dx\Big) w_2''
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
& w_1(0,t) = 0, w_1'(0,t) = 0, w_2(1,t) = 0, w_2'(1,t) = 0, \\
& w_1(\eta,t) = w_2(\eta,t), w_1'(\eta,t) = w_2'(\eta,t), \\
& w_1''(\eta,t) = w_2''(\eta,t), w_1'''(\eta,t) - w_2'''(\eta,t) = \alpha_k\, w_1(\eta,t)
\end{aligned} \tag{4}
$$

where the following abbreviations are introduced.

$$\alpha_k = k/(EI/L^3) \tag{5}$$

2.2 A systematic way

In the initial steps we tacitly used the method of multiple scales [9] to separate and obtain the linear parts of the equations in a systematic way [10]. The axial transport velocity varying about an arbitrary constant value, c_0, with a small amplitude fluctuation frequency is given as $c(t) = c_0 + \varepsilon\, c_1 \sin(\Omega t)$ where ε denotes a small variation of the velocity. The terms w_1 and w_2 representing displacements in Eq. 3 and 4 may be supposed in the form $w_1 = \sqrt{\varepsilon} y_1, w_2 = \sqrt{\varepsilon} y_2$. Thus, these terms satisfy that the longitudinal stiffness terms coming from nonlinear effects will stay in higher orders of expansions. Also, viscous damping term $\varepsilon\mu$ will be in ε order as $\bar{\mu} = \varepsilon\mu$. The displacement functions $y(x,t,\varepsilon)$ can be expressed in the series forms as follow

$$
\begin{aligned}
& y_1(x,t,\varepsilon) = y_{11}(x,T_0,T_1) + \varepsilon\, y_{12}(x,T_0,T_2) + ... \\
& y_2(x,t,\varepsilon) = y_{21}(x,T_0,T_1) + \varepsilon\, y_{22}(x,T_0,T_2) + ...
\end{aligned} \tag{6}
$$

where $T_0 = t$, is the usual fast-time scale and $T_1 = t\varepsilon$ is the slow-time scales. The time derivatives are as follow:

$$
\begin{aligned}
& d/dt = \varepsilon^0 D_0 + \varepsilon^1 D_1 + ..., \\
& d^2/dt^2 = \varepsilon^0 D_0^2 + 2\varepsilon^1 D_0 D_1 + ...
\end{aligned} \tag{7}
$$

$$\bar{v}_f^2 = \varepsilon^0\, v_f^2 \tag{8}$$

where $D_0 = \partial/\partial T_i$. Substituting expansions given with Eq. 6-8 into Eqs. 3 and 4, expanding outcomes and retaining terms up to $O(\varepsilon^0)$, result in the following equations of motion for each span. $O(\varepsilon^0)$:

$$
\begin{aligned}
& D_0^2 y_{11} + 2c_0 D_0 y_{11}' + v_f^2 y_{11}^{iv} + (c_0^2 - 1) y_{11}'' = 0, \\
& D_0^2 y_{21} + 2c_0 D_0 y_{21}' + v_f^2 y_{21}^{iv} + (c_0^2 - 1) y_{21}'' = 0
\end{aligned} \tag{9}
$$

2.3 Complex Mode Method

In this research, linear differential problem are taken into consideration, the nonlinear terms are not included in the subsequent sections. Assuming harmonic motion, Wickert and Mote [11] expressed the solution to Eq. (9) in the following span-wise characteristic functions:

$$y_{11}(x,T_0,T_1) = A(T_1)e^{i\omega T_0}W_1(x) + \overline{A}(T_1)e^{-i\omega T_0}\overline{W_1}(x)\,,$$

$$y_{21}(x,T_0,T_1) = A(T_1)e^{i\omega T_0}W_2(x) + \overline{A}(T_1)e^{-i\omega T_0}\overline{W_2}(x) \quad (10)$$

where ω represents the natural frequency, A and $\overline{A}$ are complex amplitudes and their conjugates, respectively. By means of substituting Eqs. 10 into Eqs. 9, the equations of motion can be written for each span with appropriate non-dimensional end boundary conditions and compatibility conditions at the intermediate support.

$$v_f^2 W_1^{iv} + (c_0^2 - 1)W_1'' + 2i\omega c_0 W_1' - \omega^2 W_1 = 0\,,$$

$$v_f^2 W_2^{iv} + (c_0^2 - 1)W_2'' + 2i\omega c_0 W_2' - \omega^2 W_2 = 0 \quad (11)$$

$$W_1(0,t) = 0, W_1'(0,t) = 0, W_2(1,t) = 0, W_2'(1,t) = 0\,,$$

$$W_1(\eta,t) = W_2(\eta,t), W_1'(\eta,t) = W_2'(\eta,t)\,,$$

$$W_1''(\eta,t) = W_2''(\eta,t), W_1'''(\eta,t) - W_2'''(\eta,t) = \alpha_k W_1(\eta,t) \quad (12)$$

where W_1 and W_2 is lateral displacement, ($'$) denotes differentiation with respect to the spatial variable (x), c_0 is the axial transport velocity, and v_f denotes the flexural rigidity of the beam. The deflections can be expressed in the following form [11]:

$$W_1 = d_1 e^{i\beta_5 x} + d_2 e^{i\beta_6 x} + d_3 e^{i\beta_7 x} + d_4 e^{i\beta_8 x}\,,$$

$$W_2 = d_5 e^{i\beta_5 x} + d_6 e^{i\beta_6 x} + d_7 e^{i\beta_7 x} + d_8 e^{i\beta_8 x} \quad (13)$$

where $d_1 - d_8$ denote constants. These integration constants must be determined via the boundary conditions given with Equations 12. By substituting the functions given with Eqs. 13 into Eqs. 11, the dispersion relation, $v_f^2\beta^4 + (1 - c_0^2)\beta^2 - 2\omega c_0\beta - \omega^2 = 0$, can be easily obtained. It is clear that there must be the relationships, $\beta_1 = \beta_5, \beta_2 = \beta_6, \beta_3 = \beta_7, \beta_4 = \beta_8$, among the eigenvalues. Using the boundary conditions given with Eqs. 13, the matrix of the coefficients can be obtained. We are interested in a sufficient condition for the nontriviality of the solution. The determinant of the matrix of the coefficients must vanish. Thus, the natural frequencies can be evaluated by means of Secant Method from this equality.

2.4. Stability

The beam loses its stability if the transport speed exceeds a certain value.

From the natural frequencies calculated in previous section, one can obtain eigenvalues $\mathrm{Re}(\lambda) + i\,\mathrm{Im}(\lambda)$ as complex parameter where $i\omega = \lambda$. When the real part of the eigenvalues λ is negative it indicates damping and consequently the variety of the amplitude. The imaginary parts of the eigenvalues λ characterize the natural frequencies. Depending on the signs of $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ the stability of the behaviour of the beam can be classified as [12]:

Stable: if $\mathrm{Re}(\lambda) \le 0$ (14)

Unstable with static instability (divergence): $\mathrm{Re}(\lambda) > 0$ and $\mathrm{Im}(\lambda) = 0$ (15)

Unstable with dynamic instability (flutter): $\mathrm{Re}(\lambda) > 0$ and $\mathrm{Im}(\lambda) \neq 0$ (16)

3 Divergence and critical speeds

Critical speeds are readily derived from Eqs. (16) by inserting $\omega = 0$ [12]. Therefore, after assuming the solution function $W_1(x)$ and $W_2(x)$ as an exponetial function, a dispersion relation can be obtained. From the the roots of these equations, an artificial parameter can be evaluated as follow.

$$\bar{r} = ((c_0^2 - 1)/v_f^2)^{1/2} \quad (17)$$

Using the boundary conditions given with Eq. 17, the matrix of the coefficients of the solution function can be obtained. The determinant of the matrix must vanish. After inserting v_f into the determinant, $\bar{r}$ can be evaluated numerically from this determinant by means of the Newton-Raphson Method. By substituting the calculated values of $\bar{r}$ into Eq. (18), the divergence speeds can be evaluated. The lowest divergence speed is defined as the critical speed [12-14].

$$c_{div} = \pm\sqrt{1 + \bar{r}^2\, v_f^2} \quad (18)$$

4 Numerical results

Our numerical investigations were conducted on the beam with the mechanical properties which are listed in Table 1. The flexural rigidity parameter was chosen as $v_f = 0.173$ [15].

Table 1. Mechanical properties of the system

The flexural rigidity parameter	$v_f = 0.173$
Spring-dependent coefficient	$\alpha_k = 50.118$
The location of the intermediate support	η

The first twelve speeds obtained from Eq. (18) for various η values are presented in Table 2.

Table 2. The speeds obtained from Eq. (18)

$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.3$	$\eta = 0.4$	$\eta = 0.5$
1.32137	1.39434	1.49440	1.63663	1.43254
1.79028	1.94701	2.01820	2.44162	1.84855
2.32234	2.56129	2.15527	2.86930	2.36735
2.88283	2.84184	2.87443	3.29922	2.85387
3.45796	3.20197	3.61732	4.17721	3.39312
4.04148	3.85591	3.73183	5.06501	3.90310
4.63023	4.51736	4.37189	5.51258	4.44798
5.22244	5.18345	5.13297	5.95824	4.96857
5.47091	5.49883	5.50801	6.85596	5.51531
5.81705	5.85259	5.89805	7.75347	6.04171
6.41339	6.52385	6.66576	8.20440	6.58906
7.01102	7.19663	7.30159	8.65365	7.11905

The real and imaginary parts of the complex eigenvalues versus the constant axial transport speed curves are plotted in Fig. 2 in order to show the linear instability. Depending on the signs of the real and imaginary parts of the complex eigenvalues, the boundaries of the instability regions are observed in Fig. 2.

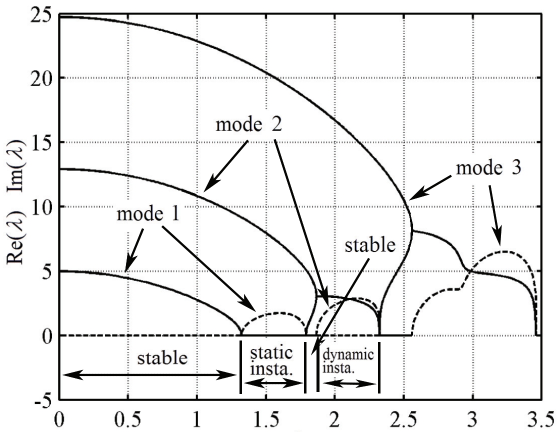

Fig. 2. The real parts (dashed line) and imaginary parts (solid line) of eigenvalues λ versus the transport speed for the first three modes

5 Summary and conclusions

The divergence speeds are analytically derived. The real and imaginary parts of the complex eigenvalues versus constant axial transport speed curves were calculated in order to show the linear instability regions. The signs of the real and imaginary parts of the eigenvalues are used to determine the boundaries of the instability regions.

Acknowledgments

This work is supported by the Scientific Research Project Unit of the Hitit University. (Project No. MUH19008.16.006). This support is gratefully acknowledged.

References

1. G. Chakraborty, A.K. Mallik, H. Hatwal, "Non-linear vibration of a travelling beam," International Journal of Non-Linear Mechanics, 4(**34**), pp.655-670, 1999.
2. K.Y. Sze, S.H. Chen, J.L. Huang, "The incremental harmonic balance method for nonlinear vibration of axially moving beams," Journal of Sound and Vibration, 3–5(**281**), pp.611-626, 2005.
3. L.H. Chen, W. Zhang, F.H. Yang, "Nonlinear dynamics of higher-dimensional system for an axially accelerating viscoelastic beam with in-plane and out-of-plane vibrations," Journal of Sound and Vibration, 25(**329**), pp.5321-5345, 2010.
4. M.H. Ghayesh, S. Balar, "Non-linear parametric vibration and stability analysis for two dynamic models of axially moving timoshenko beams," Applied Mathematical Modelling, 10(**34**), pp.2850-2859, 2010.
5. X.-D. Yang, W. Zhang, L.-Q. Chen, "Transverse vibrations and stability of axially traveling sandwich beam with soft core," Journal of Vibration and Acoustics, 5(**135**), pp.051013, 2013.
6. H.R. Öz, M. Pakdemirli, "Vibrations of an axially moving beam with time-dependent velocity," Journal of Sound and Vibration, 2(**227**), pp.239-257, 1999.
7. J.A. Wickert, "Non-linear vibration of a traveling tensioned beam," International Journal of Non-Linear Mechanics, 3(**27**), pp.503-517, 1992.
8. F. Köstekci, "Effects of flexural stiffness, axial velocity and internal support locations on translating beam's natural frequencies," Applied Mechanics and Materials, (**532**), pp.316-319, 2014.
9. A.H. Nayfeh, Introduction to perturbation techniques, Wiley, 1993.
10. M. Pakdemirli, A.H. Nayfeh, "Nonlinear vibrations of a beam-spring-mass system," Journal of Vibration and Acoustics, (**116**), pp.433-439, 1994.
11. J.A. Wickert, C.D. Mote, Jr., "Classical vibration analysis of axially moving continua," ASME Journal of Applied Mechanics, 57, pp.738–744, 1990.
12. U. Lee, Spectral element method in structural dynamics, 2009.
13. U. Lee, J. Kim, H. Oh, "Spectral analysis for the transverse vibration of an axially moving timoshenko beam," Journal of Sound and Vibration, 3-5(**271**), pp.685-703, 2004.
14. H. Oh, U. Lee, D.-H. Park, "Dynamics of an axially moving bernoulli-euler beam spectral element modeling and analysis," KSME International Journal, 3(**18**), pp.395-406, 2004.
15. M.H. Ghayesh, "Stability and bifurcations of an axially moving beam with an intermediate spring support," Nonlinear Dynamics, 1-2(**69**), pp.193-210, 2012.