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Application of Differential Transform Method and Adomian Decomposition Method for Solving of one Nonlinear Boundary-Value-Transmission Problem

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Abstract. In this study, we will found the approximate solution of one nonlinear boundary-value-transition problem by using Adomian Decomposition Method and Differential Transform Method. Namely we investigate the nonlinear differential equation,

$$y''(x) + y^2(x) = \lambda y(x), \qquad x \in [1, 2) \cup (2, 3]$$

subject to boundary conditions $y(1) = y(3) = 0$ and additional transmission conditions at the interior singular point $x = 2$, given by $y(2-0) = \gamma_1 y(2+0), \quad y'(2-0) = \gamma_2 y'(2+0)$. We obtain that using both Adomian Decomposition Method and Differential Transform Method it is possible to express analytic solutions of nonlinear boundary-value-transmission problem in terms of series without linearization, discretization or perturbation techniques.
Keywords:Adomian Decomposition Method, Differential Transform Method, approximate solution.
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INTRODUCTION

The concept of the Differential Transform Method (DTM) was first proposed by Zhou [1] for solving some initial-value problems appearing in electric circuit analysis. In the 1980's George Adomian [2] introduced a new Decomposition Method, which is an efficient method for solving linear and nonlinear ordinary differential equations, differential algebraic equations, partial differential equations, stochastic differential equations, and integral equations.

Solution using the Adomian Decomposition Method

Let us consider the nonlinear differential equation,

$$y''(x) + y^2(x) = \lambda y(x), \quad x \in [1, 2) \cup (2, 3] \tag{1}$$

subject to boundary conditions

$$y(1) = y(3) = 0 \tag{2}$$

and additional transmission conditions at the interior singular point $x = 2$, given by

$$y(2-0) = \gamma_1 y(2+0), \quad y'(2-0) = \gamma_2 y'(2+0). \tag{3}$$

By applying an our own approach, at first we will consider some auxiliary initial-value problems on the left and right side of the considered interval:

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First let us consider the auxiliary initial-value problem

$$y''(x) + y^2(x) = \lambda y(x), \qquad x \in [1,2) \tag{4}$$

$$y(1) = 0, \qquad y'(1) = a \tag{5}$$

By virtue of existence and uniqueness theorem of differential equation theory, the problem (4)-(5) has a unique solution $\widetilde{y}(x)$ [3].

By applying the decomposition method we have

$$\widetilde{y}_0(x) = ax - a,\ \ \widetilde{y}_1(x) = -\frac{1}{12}a(-2\lambda + a(-1+x))(-1+x)^3,\ \ \widetilde{y}_2(x) = \frac{a}{2520}(-1+x)^5(10a^2(-1+x)^2 - 35a(-1+x)\lambda + 21\lambda^2),$$

$$\widetilde{y}_3(x) = -\frac{a}{60480}(-1+x)^7(10a^3(-1+x)^3 - 50a^2(-1+x)^2\lambda + 63a(-1+x)\lambda^2 - 12\lambda^3), ...$$

Thus we get the fourth order approximation of the solution

$$\begin{aligned} \widetilde{y}(x) &= ax - a - \frac{1}{12}a(-2\lambda + a(-1+x))(-1+x)^3 \\ &+ \frac{a}{2520}(-1+x)^5(10a^2(-1+x)^2 - 35a(-1+x)\lambda + 21\lambda^2) \\ &- \frac{a}{60480}(-1+x)^7(10a^3(-1+x)^3 - 50a^2(-1+x)^2\lambda + 63a(-1+x)\lambda^2 - 12\lambda^3) + ... \end{aligned} \tag{6}$$

Second we will consider the right-hand problem

$$y''(x) + y^2(x) = \lambda y(x), \qquad x \in (2,3] \tag{7}$$

$$y(3) = 0, \quad y'(3) = b \tag{8}$$

We know that the problem (7)-(8) has a unique solution $\widetilde{\widetilde{y}}(x)$ (see [3]).

By using the same technique we can calculate the following first terms of the series solution $\widetilde{\widetilde{y}}(x) = \sum_{n=0}^{\infty} \widetilde{\widetilde{y}}_n(x)$, as

$$\begin{aligned} \widetilde{\widetilde{y}}(x) &= bx - 3b - \frac{1}{12}b(-3+x)^3(b(-3+x) - 2\lambda) \\ &+ \frac{b}{2520}(-3+x)^5(10b^2(-3+x)^2 - 35b(-3+x)\lambda + 21\lambda^2) + ... \end{aligned} \tag{9}$$

Using the series solutions (6)-(9) and to satisfy the transmission conditions , we must solve the following system of equations:

$$\widetilde{y}(2) = \gamma_1 \widetilde{\widetilde{y}}(2), \quad \widetilde{y}'(2) = \gamma_2 \widetilde{\widetilde{y}}'(2) \tag{10}$$

Solution using the differential transform method

Let us consider the nonlinear differential equation,

$$y''(x) + y^2(x) = \lambda y(x), \qquad x \in [1,2) \cup (2,3]$$

subject to boundary conditions,

$$y(1) = y(3) = 0$$

and additionally transmission conditions at the interior singular point $x = 2$, given by

$$y(2-0) = \gamma_1 y(2+0) \quad y'(2-0) = \gamma_2 y'(2+0).$$

First, let's get the solution for the problem in the $x \in [1,2)$. If differential transform method is applied to the differential equation,

$$Y^-(k+2) = \frac{\lambda Y^-(k) - \sum_{r=0}^{k} Y^-(r)Y^-(k-r)}{(k+2)(k+1)} \tag{11}$$

is obtained. Putting $x_0 = 1$ and using $y^-(x) = \sum_{k=0}^{n}(x-x_0)^k Y^-(k)|_{x=x_0}$, we have

$$y^-(x) = Y^-(0) + (x-1)Y^-(1) + ... + (x-1)^n Y^-(n) \tag{12}$$

By using $y(1) = 0$, the following transformed boundary condition at $x_0 = 1$ can be obtained; $Y^-(0) = 0$, $Y^-(1) = \alpha$, $Y^-(2) = 0$ where α is the unknown parameter. Following the same recursive procedure, we find $Y^-(3) = \frac{\lambda\alpha}{6}$, $Y^-(4) = \frac{-\alpha^2}{12}$, $Y^-(5) = \frac{\lambda\alpha}{120}$, $Y^-(6) = \frac{-\lambda\alpha^2}{72}$, $Y^-(7) = \frac{1}{42}\left(\frac{\lambda^2\alpha}{120} + \frac{\alpha^3}{6}\right)$, ...

Let's choose $n = 7$ in equation 12. Then, we get the following equations

$$\begin{aligned} y^-(x) &= \alpha(x-1) + \frac{\lambda\alpha}{6}(x-1)^3 - \frac{\alpha^2}{12}(x-1)^4 \\ &+ \frac{\lambda\alpha}{120}(x-1)^5 - \frac{\lambda\alpha^2}{72}(x-1)^6 + \frac{1}{42}\left(\frac{\lambda^2\alpha}{120} + \frac{\alpha^3}{6}\right)(x-1)^7 \end{aligned} \tag{13}$$

$$\begin{aligned} y'^-(x) &= \alpha + \frac{\lambda\alpha}{2}(x-1)^2 - \frac{\alpha^2}{3}(x-1)^3 \\ &+ \frac{\lambda\alpha}{24}(x-1)^4 - \frac{\lambda\alpha^2}{12}(x-1)^5 + \frac{1}{6}\left(\frac{\lambda^2\alpha}{120} + \frac{\alpha^3}{6}\right)(x-1)^6. \end{aligned} \tag{14}$$

If differential transform method is applied to the differential equation on the right interval $(2,3]$, then we have

$$Y^+(k+2) = \frac{\lambda Y^+(k) - \sum_{r=0}^{k} Y^+(r)Y^+(k-r)}{(k+2)(k+1)} \tag{15}$$

is obtained. $x_0 = 3$, using $y^+(x) = \sum_{k=0}^{n}(x-x_0)^k Y^+(k)|_{x=x_0}$,

$$y^+(x) = Y^+(0) + (x-3)Y^+(1) + ... + (x-3)^n Y^+(n) \tag{16}$$

is written. By using $y(3) = 0$, the following transformed boundary condition at $x_0 = 3$ can be obtained; $Y^+(0) = 0$, $Y^+(1) = \beta$, $Y^+(2) = 0$ where β is the unknown parameter.

Following the same recursive procedure, we find $Y^+(3) = \frac{\lambda\beta}{6}$, $Y^+(4) = \frac{-\beta^2}{12}$, $Y^+(5) = \frac{\lambda\beta}{120}$, $Y^+(6) = \frac{-\lambda\beta^2}{72}$, $Y^+(7) = \frac{1}{42}\left(\frac{\lambda^2\beta}{120} + \frac{\beta^3}{6}\right)$. In equation 16, let's choose $n = 7$. Thus, we get the following equations

$$\begin{aligned} y^+(x) &= \beta(x-3) + \frac{\lambda\beta}{6}(x-3)^3 - \frac{\beta^2}{12}(x-3)^4 \\ &+ \frac{\lambda\beta}{120}(x-3)^5 - \frac{\lambda\beta^2}{72}(x-3)^6 + \frac{1}{42}\left(\frac{\lambda^2\beta}{120} + \frac{\beta^3}{6}\right)(x-3)^7 \end{aligned} \tag{17}$$

$$\begin{aligned} y'^+(x) &= \beta + \frac{\lambda\beta}{2}(x-3)^2 - \frac{\beta^2}{3}(x-3)^3 \\ &+ \frac{\lambda\beta}{24}(x-3)^4 - \frac{\lambda\beta^2}{12}(x-3)^5 + \frac{1}{6}\left(\frac{\lambda^2\beta}{120} + \frac{\beta^3}{6}\right)(x-3)^6 \end{aligned} \tag{18}$$

Finally, substituting 17 and 18 in the transmission conditions, gives

$$\begin{aligned} &\alpha + \frac{\lambda\alpha}{6} - \frac{\alpha^2}{12} + \frac{\lambda\alpha}{120} - \frac{\lambda\alpha^2}{72} + \frac{1}{42}\left(\frac{\lambda^2\alpha}{120} + \frac{\alpha^3}{6}\right) \\ &= \gamma_1\left(-\beta - \frac{\lambda\beta}{6} - \frac{\beta^2}{12} - \frac{\lambda\beta}{120} - \frac{\lambda\beta^2}{72} - \frac{1}{42}\left(\frac{\lambda^2\beta}{120} + \frac{\beta^3}{6}\right)\right) \end{aligned} \tag{19}$$

and

$$\alpha + \frac{\lambda\alpha}{2} - \frac{\alpha^2}{3} + \frac{\lambda\alpha}{24} - \frac{\lambda\alpha^2}{12} + \frac{1}{6}\left(\frac{\lambda^2\alpha}{120} + \frac{\alpha^3}{6}\right)$$
$$= \gamma_2\left(\beta + \frac{\lambda\beta}{2} + \frac{\beta^2}{3} + \frac{\lambda\beta}{24} + \frac{\lambda\beta^2}{12} + \frac{1}{6}\left(\frac{\lambda^2\beta}{120} + \frac{\beta^3}{6}\right)\right) \tag{20}$$

CONCLUSION

The approximate solution of a considered non-linear boundary-value-transmission problem was found using differential transform method and Adomian decomposition method. The results were similar in two methods. However, in Adomian decomposition method, it was observed that it yielded more effective results with less terms. In the following table, $Dy(x)$ and $Ay(x)$ represents the result obtained with DTM and ADM respectively.

TABLE I. This table represents the results obtained with DTM and ADM.

x	Dy(x) ($\lambda = 1, \gamma_1 = 1, \gamma_2 = 1$)	Ay(x) ($\lambda = 1, \gamma_1 = 1, \gamma_2 = 1$)
1.1	0.8344762396	0.8374135822
1.2	1.6691708028	1.6750175679
1.3	2.4915688717	2.5001801538
1.4	3.2758492634	3.2868741266
1.5	3.9850957246	3.9979143730
1.6	4.5762921112	4.5900087331
1.7	5.0092610789	5.0227932802
1.8	5.2607059147	5.2730231103
1.9	5.3445151349	5.3550897237
2.1	5.3445151349	5.3550897237
2.2	5.2607059147	5.2730231103
2.3	5.0092610789	5.0227932802
2.4	4.5762921112	4.5900087331
2.5	3.9850957246	3.9979143730
2.6	3.2758492634	3.2868741266
2.7	2.4915688717	2.5001801538
2.8	1.6691708028	1.6750175679
2.9	0.8344762396	0.8374135822

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